

On Weak Tail Domination of Random Vectors

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Abstract

Motivated by a question of Krzysztof Oleszkiewicz we study a notion of weak tail domination of random vectors. We show that if the dominating random variable is sufficiently regular weak tail domination implies strong tail domination. In particular positive answer to Oleszkiewicz question would follow from the so-called Bernoulli conjecture.

Introduction. This note is motivated by the following problem about Rademacher series, posed by Krzysztof Oleszkiewicz (private communication):

Problem. Suppose that (ε_i) is a Rademacher sequence (i.e. sequence of independent symmetric ± 1 r.v.'s) and x_i, y_i are vectors in some Banach space F such that the series $\sum_i x_i \varepsilon_i$ and $\sum_i y_i \varepsilon_i$ are a.s. convergent and

$$\forall_{x^* \in F^*} \forall_{t > 0} \mathbf{P} \left(\left| x^* \left(\sum_i x_i \varepsilon_i \right) \right| \geq t \right) \leq \mathbf{P} \left(\left| x^* \left(\sum_i y_i \varepsilon_i \right) \right| \geq t \right).$$

Does it imply that

$$\mathbf{E} \left\| \sum_i x_i \varepsilon_i \right\| \leq L \mathbf{E} \left\| \sum_i y_i \varepsilon_i \right\|,$$

for some universal constant $L < \infty$?

Motivated by the above question we introduce a notion of weak tail domination of random vectors. We show that if the dominating vector has a regular distribution (including Gaussian case), weak tail domination yields strong tail domination (Theorem 1). In particular Oleszkiewicz question has positive answer provided that the so-called Bernoulli Conjecture holds true. Finally we show that in general weak tail domination does not yield comparison of means or medians of norms even if the distribution of dominated vector is Gaussian.

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Definition 1. Let X and Y be random vectors with values in some Banach space F . We say that tails of Y are weakly dominated by tails of X and denote it by $Y \prec_\omega X$ if

$$\mathbf{P}(|x^*(Y)| \geq t) \leq \mathbf{P}(|x^*(X)| \geq t) \text{ for all } x^* \in F^*, t > 0.$$

The following regularity property of random vectors will give us a tool to pass from weak to strong comparison.

Definition 2. We say that a random vector X with values in F is K -regular for some $K < \infty$ if there exists a sequence $(x_n^*) \subset F^*$ such that

$$\|x_n^*(X)\|_{\log(n+2)} = (\mathbf{E}|x_n^*(X)|^{\log(n+2)})^{1/\log(n+2)} \leq K\mathbf{E}\|X\| \text{ for } n = 1, 2, \dots$$

and

$$B_{F^*} = \{x^* \in F^* : \|x^*\| \leq 1\} \subset \text{cl}_X(\text{conv}\{\pm x_n^* : n \geq 1\}),$$

where for $A \subset F^*$, $\text{cl}_X(A)$ denotes the closure of A with respect to the L^2 distance $d_X(x^*, y^*) := (\mathbf{E}|x^*(X) - y^*(X)|^2)^{1/2}$.

Proposition 1. If X is K -regular and $Y \prec_\omega X$, then $\mathbf{E}\|Y\| \leq 20K\mathbf{E}\|X\|$.

Proof. Let x_n^* be as in Definition 2. We have for any $t > 0$,

$$\begin{aligned} \mathbf{P}\left(\sup_{n \geq 1} |x_n^*(Y)| \geq t\right) &\leq \sum_{n \geq 1} \mathbf{P}(|x_n^*(Y)| \geq t) \leq \sum_{n \geq 1} t^{-\log(n+2)} \mathbf{E}|x_n^*(Y)|^{\log(n+2)} \\ &\leq \sum_{n \geq 1} t^{-\log(n+2)} \mathbf{E}|x_n^*(X)|^{\log(n+2)} \leq \sum_{n \geq 1} \left(\frac{K\mathbf{E}\|X\|}{t}\right)^{\log(n+2)}. \end{aligned}$$

Notice that $d_Y(x^*, y^*) \leq d_X(x^*, y^*)$, hence B_{F^*} is contained also in the closure of absolute convex of $\pm x_n^*$ with respect to d_Y metric and thus

$$\begin{aligned} \mathbf{E}\|Y\| &\leq \mathbf{E} \sup_{n \geq 1} |x_n^*(Y)| \leq K\mathbf{E}\|X\| \left(e^2 + \int_{e^2}^{\infty} \mathbf{P}\left(\sup_{n \geq 1} |x_n^*(Y)| \geq tK\mathbf{E}\|X\|\right) dt \right) \\ &\leq K\mathbf{E}\|X\| \left(e^2 + \sum_{n=1}^{\infty} \int_{e^2}^{\infty} t^{-\log(n+2)} dt \right) \leq 20K\mathbf{E}\|X\|. \end{aligned}$$

□

Theorem 1. Let X_1, X_2, \dots be independent copies of symmetric random vector X . Suppose that there exist constants $K < \infty$ and $\alpha, \beta > 0$ such that for all $n = 1, 2, \dots$

- i) random vector (X_1, \dots, X_n) with values in $l_\infty^n(F)$ is K -regular,
- ii) $\mathbf{P}(\max_{i \leq n} \|X_i\| \geq \alpha \mathbf{E} \max_{i \leq n} \|X_i\|) \geq \beta$.

Then for any random vector Y such that $Y \prec_\omega X$ we have

$$\mathbf{P}(\|Y\| \geq t) \leq \frac{2}{\beta} \mathbf{P}\left(\|X\| \geq \frac{\alpha t}{80K}\right).$$

The main idea how to derive comparison of tails from comparison of means is not new - it goes back at least to the paper of de la Peña, Montgomery-Smith and Szulga [2].

Proof. We may obviously assume that Y is symmetric, by Y_1, Y_2, \dots we will denote independent copies of Y . Let $n \geq 2$ be such that

$$\frac{2}{n} \geq \mathbf{P}(\|Y\| \geq t) \geq \frac{1}{n}.$$

Then $\mathbf{P}(\max_{i \leq n} \|Y_i\| \geq t) \geq 1 - (1 - 1/n)^n \geq 1/2$, hence $\mathbf{E} \max_{i \leq n} \|Y_i\| \geq t/2$. Let η be r.v. independent of (Y_i) such that $\mathbf{P}(\eta = 1) = \mathbf{P}(\eta = 0) = 1/2$, then by Theorem 3.2.1 of [3], $\eta(Y_1, \dots, Y_n) \prec_\omega (X_1, \dots, X_n)$, where both variables are considered as random vectors in $l_\infty^n(F)$. By Proposition 1,

$$\begin{aligned} \frac{t}{4} &\leq \mathbf{E} \max_{i \leq n} \|\eta Y_i\| = \mathbf{E} \|\eta(Y_1, \dots, Y_n)\|_{l_\infty^n(F)} \leq 20K \mathbf{E} \|(X_1, \dots, X_n)\|_{l_\infty^n(F)} \\ &= 20K \mathbf{E} \max_{i \leq n} \|X_i\|. \end{aligned}$$

Property ii) yields

$$\beta \leq \mathbf{P}\left(\max_{i \leq n} \|X_i\| \geq \frac{\alpha t}{80K}\right) \leq n \mathbf{P}\left(\|X\| \geq \frac{\alpha t}{80K}\right),$$

so $\mathbf{P}(\|X\| \geq \alpha t/(80K)) \geq \beta/n \geq \beta \mathbf{P}(\|Y\| \geq t)/2$. \square

Remark 1. By the Paley-Zygmund inequality (cf. [3], Lemma 0.2.1), the comparison of first and second moments of maxima,

$$\mathbf{E} \max_{i \leq n} \|X_i\|^2 \leq C(\mathbf{E} \max_{i \leq n} \|X_i\|)^2 \quad (1)$$

implies property ii) of previous theorem with $\alpha = 1/2$ and $\beta = 1/(4C)$.

Remark 2. Both Proposition 1 and Theorem 1 hold (with constants depending on C_1 and C_2) if we replace the condition $Y \prec_\omega X$ by the condition

$$\mathbf{P}(|x^*(Y)| \geq t) \leq C_1 \mathbf{P}(|x^*(X)| \geq t/C_2) \text{ for all } x^* \in F^*, t > 0. \quad (2)$$

Indeed, if η is a random variable independent of Y with $\mathbf{P}(\eta = 1) = 1 - \mathbf{P}(\eta = 0) = 1/C_1$, then condition (2) implies $\eta Y/C_2 \prec_\omega X$.

Let us give few examples of random vectors satisfying the assumptions of Theorem 1.

1. Any centered Gaussian vector on a separable Banach space is L -regular with universal L . This is a consequence of majorizing measure theorem (cf.[5] and [6], Theorem 2.1.8). Since a product of Gaussian measures is again Gaussian, property i) holds with $K = L$. Moments of Gaussian vectors are comparable so by Remark 1 also property ii) holds with $\alpha = 1/2$ and universal β .

2. Let (η_i) be a sequence of independent symmetric real r.v.'s with logarithmically concave tails satisfying Δ_2 condition and $v_i \in F$ be such that $X = \sum_i v_i \eta_i$ is a.s. convergent. Then X is K -regular with constant K depending only on Δ_2 constant ([4], Theorem 3). Random variable (X_1, \dots, X_n) has an analogous series representation in $l_\infty^n(F)$, so property i) holds. It can be also checked that (1) is satisfied with universal C .

3. Positive answer to Bernoulli Conjecture ([6], Chapter 4) would imply the L -regularity of Rademacher series. Since (1) holds for X being a Rademacher sum with vector coefficients, Theorem 1 would give positive answer to Oleszkiewicz question.

We conclude with an example showing that weak tail domination does not yield any comparison of strong parameters even if the dominated vector has Gaussian distribution.

Example. Let $F = l_\infty^n$, $Y = \sum_{i=1}^n g_i e_i$ and $X = 9(|g_1| + 1) \sum_{i=1}^n \eta_i e_i$, where g_i are i.i.d. $\mathcal{N}(0, 1)$ and η_i are i.i.d. r.v.'s with uniform distribution on $[-1, 1]$, independent of g_1 .

To show that tails of Y are weakly dominated by tails of X it is enough to check that

$$\mathbf{P}(|\langle u, Y \rangle| \geq t) \leq \mathbf{P}(|\langle u, X \rangle| \geq t) \text{ for } u \in S^{n-1}, t \geq 0. \quad (3)$$

Let us fix $u \in S^{n-1}$. For any $t > 0$ we have

$$\mathbf{P}(|\langle u, Y \rangle| \geq t) = \mathbf{P}(|g_1| \geq t).$$

By the Paley-Zygmund inequality,

$$\begin{aligned} \mathbf{P}\left(\left|\sum_{i=1}^n u_i \eta_i\right| \geq \frac{1}{3}\right) &= \mathbf{P}\left(\left|\sum_{i=1}^n u_i \eta_i\right|^2 \geq \frac{1}{3} \mathbf{E}\left|\sum_{i=1}^n u_i \eta_i\right|^2\right) \\ &\geq \left(1 - \frac{1}{3}\right)^2 \frac{(\mathbf{E}|\sum_{i=1}^n u_i \eta_i|^2)^2}{\mathbf{E}|\sum_{i=1}^n u_i \eta_i|^4} \geq \frac{4}{27}, \end{aligned}$$

thus

$$\mathbf{P}(|\langle u, X \rangle| \geq t) \geq \frac{4}{27} \mathbf{P}(3(|g_1| + 1) \geq t) \geq \frac{4}{27} \mathbf{P}\left(|g_1| \geq \frac{t}{3}\right).$$

Using the simple estimate $2t \exp(-(2t)^2/2)/\sqrt{2\pi} \leq \mathbf{P}(|g| \geq t) \leq \exp(-t^2/2)$, we immediately get (3) for $t \geq 3$. For $0 \leq t \leq 3$ we have

$$\begin{aligned} \mathbf{P}(|\langle u, X \rangle| \leq t) &\leq \mathbf{P}\left(9\left|\sum_{i=1}^n u_i \eta_i\right| \leq t\right) \leq \frac{\sqrt{2}t}{9} \leq t \frac{\mathbf{P}(|g_1| \leq 3)}{3} \\ &\leq \mathbf{P}(|g_1| \leq t) = \mathbf{P}(|\langle u, Y \rangle| \leq t), \end{aligned}$$

where to get the second inequality we used Ball's upper bound on cube sections [1]. Hence (3) holds also for $t \in [0, 3]$.

Thus $Y \prec_\omega Y$. However $\mathbf{E}\|Y\| = \mathbf{E} \max_{i \leq n} |g_i| \geq \sqrt{\log n}/L$ and $\mathbf{E}\|X\| \leq 9\mathbf{E}(|g_1| + 1) \leq 18$.

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